CORRIGENDUM TO "THE INTERSECTION MOTIVE ON THE MODULI STACK OF SHTUKAS"

TIMO RICHARZ AND JAKOB SCHOLBACH

Abstract. We correct the statements of Proposition 3.1.23 and Theorem 2.1.15 in [RS20]. None of the results in [RS20, §§4–6] and [RS21] are affected by these corrections.

1.1. **Around** [RS20, Proposition 3.1.23]**.** The proposition needs to be replaced as follows.

Proposition 1.1. *Let $H \subset G$ be an inclusion of ordinary τ-sheaves of S-groups where τ is a Grothendieck topology as in* [RS20, Theorem 2.2.16]*. Let $X := (G/H)^\tau$ be the quotient of τ-sheaves which we assume to be a smooth finite type S-scheme. Let $p : G \to X$ be the quotient map. Then, the equivalence $\mathrm{DM}_G(X) = \mathrm{DM}_H(S)$ restricts to an inclusion of full subcategories*

$$\mathrm{DTM}_G(X) \subset \mathrm{DTM}_H(S).$$

This inclusion is an equivalence if $p^!$ detects Tate motives, that is, if $p^! M \in \mathrm{DTM}(G)$, then $M \in \mathrm{DTM}(X)$.

Proof. Without comment, we use the τ-descent equivalence $\mathrm{DM}(X) = \mathrm{DM}(G/H)$. We consider the following augmented, doubly cosimplicial diagram

$$\begin{array}{ccccc}
G \times G \times H & \rightrightarrows & G \times H & \longrightarrow & G \setminus G \times H \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow \\
G \times G & \rightrightarrows & G & \longrightarrow & G \setminus G = S \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
G \times G/H & \longrightarrow & G/H & \longrightarrow & G \setminus G/H.
\end{array}$$

By construction, a motive $M \in \mathrm{DM}_G(X) = \mathrm{DM}(G \setminus G/H) = \mathrm{DM}(S/H)$ is a family of motives $M_{n,m} \in \mathrm{DM}(G^n \times G \times H^m)$, compatible under !-pullback along the various action and projection maps.

By definition, $M \in \mathrm{DTM}_H(S)$ if and only if M is a Tate motive on $S = G \setminus G$. Using the section of the structural map $G \to S$, this is equivalent to requiring $M_{0,0} \in \mathrm{DTM}(G)$, or equivalently all $M_{n,m} \in \mathrm{DTM}(G^n \times G \times H^m)$. By comparison, $M \in \mathrm{DTM}_G(X)$ amounts to requiring that M restricts to a Tate motive on X. □

Remark 1.2. A counter-example to the assertion in [RS20, Proposition 3.1.23] is the case $H = \mu_2 \subset G = \mathbf{G}_\mathrm{m}$. The proof fails because diagram (3.1.24) does not commute (only if the map β there is reversed).

Date: 20/12/2023.

The additional assumption about detection of Tate motives under $p^!$ also applies to the subsequent statements [RS20, Proposition 3.2.22, 3.2.23].

We now give a sufficient criterion for the map p to detect Tate motives which is satisfied in the context of affine flag varieties in [RS20] and their Witt vector versions in [RS21]. In view of the following results, all results in [RS20, §§4–6] and [RS21] do hold as stated.

Proposition 1.3. *In the situation of Proposition 1.1, suppose H is an extension of a split reductive S-group scheme by a split unipotent S-group scheme U in the sense of* [RS20, Definition A.4.5]*, that is, U is a successive extension of vector S-group schemes. Suppose $p : G \to X$ is an H-torsor between smooth S-schemes of finite type that is trivial on an open, fiberwise dense subset of X. Then, both functors $p^!$ and p^* detect Tate motives.*

Proof. By smoothness, $p^!$ and p^* agree up to shift and twist, so it is enough to treat the case p^*. For $M \in \mathrm{DM}(X)$ we have the projection formula $p_! p^* M = p_! 1_G \otimes M$. By Proposition 1.4, $p_! 1_G$ is a Tate motive on X, including $1_X(-d_H)[-2d_H]$ as a direct summand where d_H denotes the relative dimension of H over S. □

Proposition 1.4. *Let H be as in Proposition 1.3. Let $f : Y \to X$ be an H-torsor between smooth S-schemes of finite type that is trivial on an open, fiberwise dense subset $V \subset X$. Then, $f_! 1_Y$ and $f_* 1_Y$ are Tate motives, that is, objects in* $\mathrm{DTM}(X)$.

Proof. By duality, it suffices to consider $f_!$. We may assume S and X are connected. We have the geometric quotients $Y \overset{a}{\to} Y/U \to Y/H = X$. The map a is a successive extension of affine bundles, so $a_! a^*$ is, up to twist and shift, the identity functor. We may therefore assume H to be split reductive.

Let $T \subset B \subset H$ be a split maximal torus and a Borel subgroup. We consider the geometric quotients

$$Y \overset{t}{\to} Y/T \overset{b}{\to} Y/B \overset{p}{\to} Y/H = X.$$

We claim that all !-pushforwards preserve Tate motives. It is clear for b because B/T is split unipotent. Since T is split, the map t is a composition of $\mathbf{G}_{\mathrm{m}}$-torsors. Every $\mathbf{G}_{\mathrm{m}}$-torsor admits étale local sections and thus is Zariski-locally trivial by Hilbert's Theorem 90. Thus, $t_!$ preserves Tate motives by the computations as in [HK06, Theorem 8.8]. It remains to consider the map p that has general fiber $F := H/B$ and is trivial over the open, fiberwise dense subset $V \subset X$.

The fiber F has a stratification by affine spaces given by the Schubert cells. So F satisfies Poincaré duality, that is, the pairing $\mathrm{CH}_p(F) \otimes \mathrm{CH}^p(F) \to \mathrm{CH}_0(F) \cong \mathbf{Z}$ is perfect. Indeed, these Chow groups are free abelian groups of finite rank (indexed by the strata) and the assertion is obvious. Recall from [RS20, Synopsis 2.1.1] that for any regular, finite-type S-scheme Z, we have an identification $\mathrm{Hom}_{\mathrm{DM}(Z)}(1_Z, 1_Z(p)[2p]) = K_0(Z)^{(p)}_{\mathbf{Q}}$ with the Adams eigenspace, which in turn is isomorphic to $\mathrm{CH}^p(Z)_{\mathbf{Q}}$, see [Sta17, Tag 0FEW]. Note that our running assumptions in [RS20, Notation 2.0.1] ensure that S satisfies [Sta17, Tag 0F91].

By assumption, the base change $Y_V := Y \times_X V$ is isomorphic to $F \times_S V$. Since F has a stratification by affine spaces, we have an isomorphism $\mathrm{CH}^*(Y_V) = \mathrm{CH}^*(F) \otimes_{\mathbf{Z}} \mathrm{CH}^*(V)$. Pick a basis $b_1, \dots, b_n$ of $\mathrm{CH}^*(F)$, and choose representatives $b_r = \sum_s a_{rs}[V_{rs}]$ for certain cycles $V_{rs} \subset F$. We consider the elements

$\bar{b}_r := \sum_s a_{rs}[\overline{V_{rs} \times_S V}] \in \mathrm{CH}^*(Y)$. These give rise to a morphism

$$\alpha : f_\sharp 1_Y \to \bigoplus_p \mathrm{CH}_p(F) \otimes 1_X(p)[2p].$$

We claim that α is an isomorphism in $\mathrm{DM}(X)$. By localization, it suffices to show its pulback along any point $\operatorname{Spec} k(s) \to S$ is an isomorphism, so we may assume S is a field.

As the map $f : Y \to X$ is étale-locally on X equal to the projection $F \times X \to X$ and DM satisfies étale descent, we are reduced to proving the following statement: if $f : Y = F \times X \to X$ is the projection and α as above is a map whose restriction to an open, dense subscheme $V \subset X$ is an isomorphism, then α itself is an isomorphism. Given the stratification of F by affine spaces, we see that $f_\# 1_Y$ is a direct sum of motives of the form $1(n)[2n]$. Therefore our claim follows from Lemma 1.5 below, applied to the generic point of X (or any point in V). □

Lemma 1.5. *Let X be a smooth connected scheme over a field, and $x \in X$ some point. Let $M \in \mathrm{DM}(X)$ be a finite direct sum of the form $\bigoplus_{n \in \mathbf{Z}} 1_X(n)[2n]^{m_n} \in \mathrm{DM}(X)$, and let $f : M \to M$ be an endomorphism in $\mathrm{DM}(X)$. Then f is an isomorphism if and only if $x^* f$ is an isomorphism.*

Proof. We have $\mathrm{Hom}_{\mathrm{DM}(X)}(1, 1(n)[2n]) = 0$ for $n < 0$. Therefore, f is an isomorphism if and only if all its "diagonal components", i.e., the composites

$$1(n)[2n]^{m_n} \subset M \xrightarrow{f} M \xrightarrow{\mathrm{pr}} 1(n)[2n]^{m_n}$$

are isomorphisms, and likewise for $x^* f$. We conclude using that pullback along $\operatorname{Spec} k(x) \to X$ induces an isomorphism on $\mathrm{End}_{\mathrm{DM}(X)}(1(n)[2n])$ as X is connected. □

Remark 1.6. For motivic sheaves with integral coefficients that only satisfy Zariski (or Nisnevich, as opposed to étale) descent, Proposition 1.4 remains true if the H-torsor is Zariski (or Nisnevich) locally trivial.

Remark 1.7. Proposition 1.4 relates to several results in the literature. If $S = \operatorname{Spec} k$ is a field, Edidin–Graham [EG97, Proposition 1] construct a (non-canonical) isomorphism of Chow rings $\mathrm{CH}^*(Y) \simeq \mathrm{CH}^*(F) \otimes \mathrm{CH}^*(X)$. If X is stratified by affine spaces, which is the case for the applications in [RS20, RS21], such an isomorphism is proved by Rossello–Xambó-Descamps [RLXD88, Theorem 2]. Using [EG97] Arasteh–Habibi [AH17, Theorem 2.8] shows that the motive $\mathrm{M}(Y) \in \mathrm{DM}(S)$ is a direct sum of twisted and shifted copies of $\mathrm{M}(X)$. Proposition 1.4 extends that statement to general base schemes S and shows Tateness within $\mathrm{DM}(X)$, as opposed to a computation in $\mathrm{DM}(S)$.

1.2. **Around** [RS20, Theorem 2.1.15]**.** Lemma 2.1.16 is false and needs to be erased. It is only used in the proof of Theorem 2.1.15 whose statement needs to be replaced by the following one.

Theorem 1.8. *Suppose S satisfies the assumptions in Notation 2.0.1. Then, for $\Lambda \in \{\mathbf{Q}_\ell, \mathbf{Z}_\ell, \mathbf{Z}/\ell^n\}$ the presheaf*

$$\mathrm{D}_{\mathrm{cons}}(-, \Lambda) : (\mathrm{Sch}_S^{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Cat}_\infty, X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda), f \mapsto f^!$$

is a sheaf in the h-topology. If all finite type S-schemes have finite $\mathbf{Z}/\ell$-cohomological dimension as in [HRS23, Section 8]*, then $\mathrm{D}_{\text{ét}}(-, \Lambda) := \operatorname{Ind} \mathrm{D}_{\mathrm{cons}}(-, \Lambda)$ taking values in Cat_∞ is an h-sheaf as well.*

Proof. The proof for $\mathrm{D}_{\mathrm{cons}}(-,\Lambda)$ is the same as in Theorem 2.1.15. As for $\mathrm{D}_{\text{ét}}(-,\Lambda)$, we prove the conservativity of $f^!$ for an arbitrary h-cover $f\colon Y\to X$ in $\mathrm{Sch}_S^{\mathrm{ft}}$ by replacing Lemma 2.1.16 by the additional hypothesis on the cohomological dimension. Under that hypothesis, the category $\mathrm{D}_{\text{ét}}(X,\Lambda)$ is a full subcategory of $\mathrm{D}(X_{\text{proét}},\Lambda)$ by [HRS23, Corollary 8.3] compatibly with the pullback functor which can be shown to be conservative. □

Remark 1.9. The categories $\mathrm{D}_{\mathrm{cons}}(-,\Lambda)$ defined as in [HRS23] even satisfy descent for the arc topology by [HS23]. The finiteness assumption on the $\mathbf{Z}/\ell$-cohomological dimension is satisfied for $S=\operatorname{Spec}\mathbf{F}_p$, for example, which is the case relevant for [RS20].

References

[AH17] Rad Esmail Arasteh and Somayeh Habibi. On the motive of a fibre bundle and its applications. *Anal. Geom. Number Theory*, 2:77–96, 2017. doi:10.19272/201712501004. 3

[EG97] Dan Edidin and William Graham. Characteristic classes in the Chow ring. *J. Algebr. Geom.*, 6(3):431–443, 1997. 3

[HK06] Annette Huber and Bruno Kahn. The slice filtration and mixed Tate motives. *Compos. Math.*, 142(4):907–936, 2006. 2

[HRS23] Tamir Hemo, Timo Richarz, and Jakob Scholbach. Constructible sheaves on schemes. *Adv. Math.*, 429:46, 2023. Id/No 109179. arXiv:2305.18131, doi:10.1016/j.aim.2023.109179. 3, 4

[HS23] David Hansen and Peter Scholze. Relative perversity. *Comm. Amer. Math. Soc.*, 3:631–668, 2023. doi:10.1090/cams/21. 4

[RLXD88] Francesco Rosselló Llompart and Sebastian Xambó-Descamps. Computing Chow groups. Algebraic geometry, Proc. Conf., Sundance/Utah 1986, Lect. Notes Math. 1311, 220-234 (1988)., 1988. 3

[RS20] Timo Richarz and Jakob Scholbach. The intersection motive of the moduli stack of shtukas. *Forum of Mathematics (Sigma)*, 8(e8), 2020. URL: https://arxiv.org/abs/1901.04919, doi:10.1017/fms.2019.32. 1, 2, 3, 4

[RS21] Timo Richarz and Jakob Scholbach. Tate motives on Witt vector affine flag varieties. *Selecta Math.*, 2021. URL: https://arxiv.org/abs/2003.02593, doi:10.1007/s00029-021-00665-y. 2, 3

[Sta17] The Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu, 2017. 2